Section 4: Random variables – continuous

STA 35C – Statistical Data Science III

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MWF, 12:10 PM – 1:00 PM, Olson 158
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Overview

Based on Chapter 4 of textbook: `https://www.probabilitycourse.com/`

- Contains problems with solutions, and problems without solutions.

Introduction

Introduction

- A *discrete* random variable has a countable range.
- A *continuous* random variable has a range in the form of an interval or a union of non-overlapping intervals on the real line (possibly the whole real line).
  For any $a \in \mathbb{R}$, we have $P(X = a) = 0$.
- Need new tools. Overview: replace sums with integrals; replace PMFs with PDFs.

Example

Find the CDF of a real number chosen *uniformly at random* in the interval $[a, b]$.

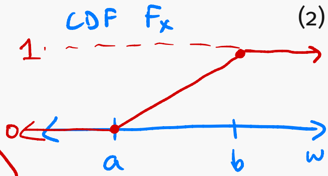
- *Uniformly at random*: all intervals in $[a, b]$ that have the same length must have the same probability.
- For any interval subset $[c, d]$ of $[a, b]$ with $c < d$, uniformity assumption implies

$$P(X \in [c, d]) \propto (d - c). \tag{1}$$

"proportional to"

- Since $P(X \in [a, b]) = 1$, we conclude

$$P(X \in [c, d]) = \frac{d - c}{b - a}. \tag{2}$$



To get the CDF, note that

- $F_X(w) = 0$ if $w < a$;
- $F_X(w) = 1$ if $w > b$;
- For $w \in [a, b]$, we have

$$F_X(w) = P(X \leq w) = P(X \in [a, w]) = \frac{w - a}{b - a}. \tag{3}$$

Thus the CDF looks like (no jumps)

Definition

In the previous example, the CDF is a continuous function (it has no jumps).

- Recall: jumps in the CDF correspond to values a for which $P(X = a) > 0$.
- "No jumps" is consistent with $P(X = a) = 0$ for all a.

Definition 1: continuous random variable

A random variable X is said to be *continuous* if its CDF $F_X\colon \mathbb{R} \to [0, 1]$ is continuous.

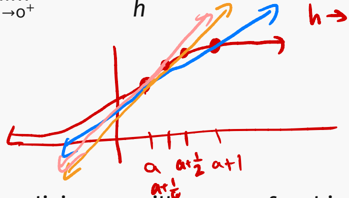
Probability density function

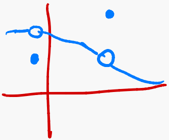
Motivation

To determine the distribution of a random variable of any type, we can provide its CDF.

- For a *continuous* r.v., we cannot use a PMF because $P(X = a) = 0$ for all $a \in \mathbb{R}$.
- We will instead define its *probability density function (PDF)*.
  Similar to mass density in physics.
- For a continuous r.v. X and a value $a \in \mathbb{R}$, consider the quantity

$$\lim_{h \to 0^+} \frac{P(a < X \le a + h)}{h} = \lim_{h \to 0^+} \frac{F_X(a+h) - F_X(a)}{h} = F_X'(a)$$





For simplicity, we will assume for this course that the CDF of a continuous random variable is differentiable *except at possibly a few values in* $\mathbb{R}$.

- "almost everywhere" in $\mathbb{R}$

Definition

Definition 2: probability density function (PDF)

Consider a continuous random variable X with an "absolutely continuous" CDF F_X. The function f_X defined by

$$f_X(a) = \frac{\mathrm{d}F_X(a)}{\mathrm{d}a} = F_X'(a), \quad \text{if } F_X(a) \text{ is differentiable at } a \tag{4}$$

is called the *probability density function (PDF)* of X.

"absolutely continuous" allows us to obtain the CDF from a PDF (we'll see this later)

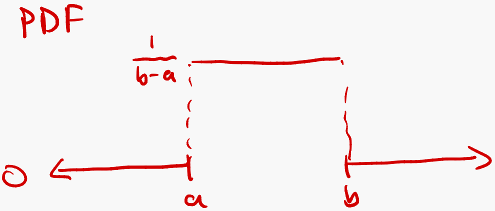
Example

Let's find the PDF of the uniform random variable discussed in slide 3.

- This random variable is said to have a *Uniform*(a, b) distribution.
- The CDF of X is given in slide 3. By taking the derivative, we obtain the PDF:

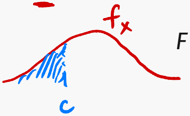
$$f_X(c) = \begin{cases} \frac{1}{b-a} & \text{if } a < c < b, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

This function (the PDF) looks like



Obtain the CDF from a PDF

If we are given a PDF f_X, then the CDF is the function F defined by



$$F(c) = \int_{-\infty}^{c} f_X(u)\,\mathrm{d}u \tag{6}$$

- "absolutely continuous" is a stronger condition than "continuous"; allows us to use the fundamental theorem of calculus.
- For this course, assume that all CDFs of *continuous* r.v.s are absolutely continuous.

From Eq. (6) we also get F is the CDF of X

(6)

$$P(a < X \le b) = F(b) - F(a) = \int_{a}^{b} f_X(u)\,\mathrm{d}u \tag{7}$$

In particular, since $P(-\infty < X \le \infty) = 1$, we must also have

$$\int_{-\infty}^{\infty} f_X(u)\,\mathrm{d}u = 1, \tag{8}$$

i.e., *the area under the PDF curve must be equal to one.*

Summary of properties of a PDF

Consider a continuous random variable X with PDF f_X. We have

1. $f_X(c) \geq 0$ for all $c \in \mathbb{R}$.
2. Eq. (8) — area under PDF curve = 1
3. Eq. (7)
4. More generally, for a set A, we have $P(X \in A) = \int_A f_X(u)\,\mathrm{d}u$.

For item 4, the set A must satisfy some mild conditions which are almost always satisfied in practice. E.g., the set A could be a union of some disjoint intervals:

$$P\Big(X \in \overbrace{[0,2] \cup [3,4]}\Big) = \underbrace{\int_0^2 f_X(u)\,\mathrm{d}u} + \underbrace{\int_3^4 f_X(u)\,\mathrm{d}u}.$$

Example

Let X be a continuous random variable with the following PDF

$$f_X(u) = \begin{cases} c\,e^{-u} & u \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant.

1. Find c.
2. Find the CDF of X.
3. Find $P(1 < X < 3)$.

$$\boxed{1.\quad 1 = \int_{-\infty}^{\infty} f_X(u)\,du = \int_0^{\infty} c\,e^{-u}\,du \quad \Rightarrow c = \frac{1}{\int_0^{\infty} e^{-u}\,du} = \frac{1}{1} = 1}$$

$$\boxed{2.\quad F_X(b) = \int_{-\infty}^{b} f_X(u)\,du = \begin{cases} 0 & b < 0 \\ \int_0^b e^{-u}\,du = 1 - e^{-b}, & b \geq 0 \end{cases}}$$

$$3.\quad P(1 < X < 3) = F_X(3) - F_X(1) = e^{-1} - e^{-3}.$$

$$P(1 \leq X \leq 3) = \underbrace{P(1 = X)}_{0} + P(1 < X < 3) + \underbrace{P(X = 3)}_{0}$$

If X is a continuous random variable, we can define the *range of X* as

$$R_X = \{c \mid f_X(c) > 0\},$$

i.e., the set of real numbers c for which the PDF f_X is larger than zero.

Expected value and variance

Definition

The *expected value* of a random variable is the weighted average of all of its values, where the weights are the probabilities that these values occur.

Definition 3: Expected value $E(\cdot)$

Let X be a continuous random variable. Then the *expected value* of X is defined as

$$E(X) = \int_{-\infty}^{\infty} u \, f_X(u) \, \mathrm{d}u \tag{9}$$

Expected Value of a Function of a Continuous Random Variable

Law of the unconscious statistician (LOTUS) for continuous random variables

$$E(g(X)) = \int_{-\infty}^{\infty} g(u)\, f_X(u)\, \mathrm{d}u \tag{10}$$

The "Expectation is linear" properties from slide 21 of Sec 3 also hold for continuous random variables.

Variance

Same expressions as with discrete random variables, but this time using Eq. (9) for the expected value:

$$Var(X) = E\left[(X - EX)^2\right] = E\left[X^2\right] - (EX)^2 \tag{11}$$

- The variance properties in slides 28 and 29 of Sec 3 also hold for continuous random variables.

Special Distributions

A random variable X is *uniformly* distributed on an interval $M = (a, b)$, with $b > a$, if the PDF has the form

$$f_X(x) = \frac{1}{b-a} \quad \text{for all } x \in (a, b).$$

- Such distributions occur when all (uncountably many) possible outcomes are equally likely.
- The interval M can also instead be $[a, b)$, or $(a, b]$, or $[a, b]$.
- Here we also write $X \sim U(M)$ or $X \sim Unif(M)$.
- Nine random draws in $(3, 5)$ in R: `runif(n=9, min=3, max=5)`
- Expected value and variance

$$EX = \frac{a+b}{2} \qquad \text{and} \qquad Var(X) = \frac{(b-a)^2}{12}.$$

End of 10/13 lecture

A random variable X is *normally* distributed with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, if the PDF has the form

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for all } x \in \mathbb{R}.$$

- This distribution appears often in this class, in future classes, and in life!
- We write $X \sim N(\mu, \sigma^2)$. We also call it *Gaussian* distributed.
- Expected value (location parameter) and variance (squared scale parameter)

$$E(X) = \mu \quad \text{and} \quad Var(X) = \sigma^2.$$

- If $X \sim N(0, 1)$, the distribution of X is said to be *standard normal*.
- Nine random draws in R: `rnorm(n=9, mean=2, sd=1)`

PDF of $X \sim N(0,1)$, $Y \sim N(2,1)$, $Z \sim N(0,3)$