Section 1: Basic concepts in probability

STA 35C – Statistical Data Science III

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MWF, 12:10 PM – 1:00 PM, Olson 158
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Section 1: Overview

Based on Chapter 1 of textbook: `https://www.probabilitycourse.com/`

- Section 1.1: Introduction
- Section 1.2: Review of Set Theory
- Section 1.3: Random Experiments

Section 1: Basics in probability theory

Section 1.1: Introduction

What is probability?

Probability theory is a way to quantify *randomness* and/or *uncertainty*.

- "Randomness" is a type of uncertainty, but uncertainty encompasses more.
- Aleatoric uncertainty vs epistemic uncertainty.
- Game of 20 questions.
- Possible confusion: in the literature, "random" sometimes describes uncertainty. E.g., random experiment, random variable

What is probability?

Consider a simple possible random experiment: flipping a "fair" coin.

- What is a "fair" coin? Probability of heads is 1/2, but what does this mean?
- One interpretation is *relative frequency*: if we flip the coin many times, it will come up heads about 1/2 of the time.
- As # of coin flips increases, the *proportion* that come up heads will tend to get closer and closer to 1/2.

$$\text{relative frequency} = \frac{\#\ \text{heads}}{\#\ \text{flips}}$$

What is probability?

Another interpretation of probability is *personal belief* that something will happen.

- Consider another example: what is the probability that it will rain later today?
- We can consider e.g. humidity and whether there are clouds in the sky.
- Even if two people observe the same things, their resulting beliefs can still differ: different people may make different estimates of the probability that it will rain.
- This is because two people can have different *prior* beliefs, i.e., beliefs before observing the data.
- If two people have the same prior beliefs and observe the same data, then their updated beliefs should be the same if they use the same updating mechanism.

What is probability?

Often, these two interpretations of probability coincide.

- E.g. we may base our *personal beliefs* on an assessment of the *relative frequency* of rain on days with conditions like today.

Probability theory is applicable *regardless of the interpretation of probability* (e.g., relative frequency, subjective personal belief) that we use.

- Starts by assuming axioms of probability, and then building the entire theory using mathematical arguments.
- Probability theory uses the language of *sets*.

Section 1: Basics in probability theory

Section 1.2: Review of Set Theory

A *set* is a collection of some "elements" (can also call them "items" or "members"). Can write by placing elements in curly braces $\{\}$. Examples:

$\{1, 3\}$ $B = \{\heartsuit, \spadesuit, \diamondsuit\}$

$A = \{\spadesuit, \diamondsuit\}$ $\Rightarrow A \subset B$

$\in$ = "belongs to"

Notation

- To say that an item $\heartsuit$ *belongs to* a set A, we write $\heartsuit \in A$.
- To say that an item $\diamondsuit$ *does not belong to* a set A, we write $\diamondsuit \notin A$.
- Set A is a *subset* of set B if every element of A is also an element of B. Write $A \subset B$. We can also say B is a *superset* of A, or $B \supset A$.
- Set A is *equal* to set B if both $A \subset B$ and $B \subset A$. Write $A = B$. Ordering does not matter: the two sets $\{\heartsuit, \diamondsuit\}$ and $\{\diamondsuit, \heartsuit\}$ are the same.

Important example sets

- The set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$
- The set of integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ $\subseteq$
- The set of real numbers, $\mathbb{R}$
- Closed intervals on the real line, e.g., $[2, 3]$ $\subseteq$
- Open intervals on the real line, e.g., $(-1, 3)$
- Half-closed (half-open) intervals on the real line, e.g., $[1, 2)$
- The set with no elements, i.e., the *empty set* $\varnothing$

Exercise

1. What sets above are supersets of $\mathbb{N}$?
2. What sets above are subsets of $(-1, 3)$?

Another way to define a set

We can also define a set by stating the properties satisfied by the elements in the set:

$$A = \{\text{function of } x \mid x \text{ satisfies some property}\}.$$

Equivalently written as

$$A = \{\text{function of } x : x \text{ satisfies some property}\}.$$

The symbols "|" and ":" are pronounced "such that."

Exercise: what is the set (and how do you say it)

1. $\{x + 2 \mid x \in (3,5)\}$? $= (5,7)$
2. $\{x \mid x \in \mathbb{Z}, -2 \leq x < 10\}$? $= \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
3. $\{x^2 \mid x \in \mathbb{N}\}$? $= \{1, 4, 9, 16, 25, \ldots\}$
4. $\left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\}$?

= set of rational numbers

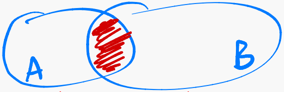
9 16 25

$3^2 + 4^2 = 5^2$

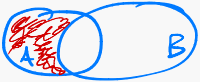
Set operations

The *union* $A \cup B$ of two sets A and B:
the set consisting of all elements that are in A or in B (possibly both).

The *intersection* $A \cap B$ of two sets A and B:
the set consisting of all elements that are both in A and B.



The *difference (subtraction)* $A \setminus B$ of two sets A and B:
the set consisting of all elements that are in A but not in B.



The *complement* A^c or $\bar{A}$ of a set A: $A^c = S \setminus A$
the set of all elements that are in the *universal set* S but are not in A.

The *Cartesian product* $A \times B$ of two sets A and B: the set consisting of ordered pairs from A and B. Mathematically:

for A   from B

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{H, T\}$, then $A \times B$ is the set

$$\{(1, H), (1, T), (2, H), (2, T), (3, H), (3, T)\}.$$

- Here the pairs are ordered, e.g., $(1, H) \neq (H, 1)$.
- Thus $A \times B$ is not the same as $B \times A$.

(Named after René Descartes (1596-1650), best known for "I think, therefore I am.")

Set operations: examples

Exercise: If the universal set is given by $S = \{1, 2, 3, 4, 5, 6\}$, and $A = \{1, 2\}$, $B = \{2, 4, 5\}$ are two sets, find the following sets:

- $\bar{A} = \{3, 4, 5, 6\}$
- $\bar{B} = \{1, 3, 6\}$
- $B \setminus A = \{4, 5\}$
- $A \times B = \{(1, 2), (1, 4), (1, 5)$
  $(2, 2), (2, 4), (2, 5)\}$

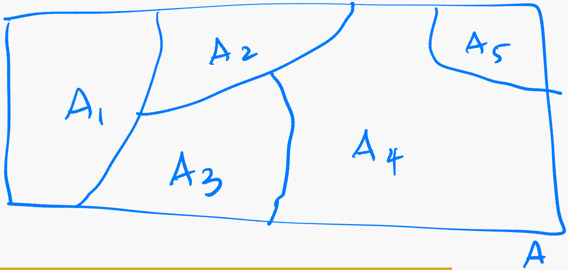
Two sets A and B are *mutually exclusive* or *disjoint* if they share no elements: $A \cap B = \varnothing$.

$A = (1, 2)$

$B = (2, 4)$

A collection of nonempty sets $A_1, A_2, \ldots$ is a *partition* of a set A if they are disjoint and their union is A.

$C = \{2\}$



$A \cup B \cup C = (1, 4)$

end of 9/24 lecture

Cardinality of infinite sets

The *cardinality* of a set A is basically the size of the set. It is denoted by $|A|$.

Distinguish between two types of infinite sets: countable and uncountable. Examples:

- Sets such as $\mathbb{N}$ and $\mathbb{Z}$ (and any of their subsets) are called *countable*. A finite set is countable.
- "Bigger" sets such as $\mathbb{R}$ are *uncountable*.
  Intervals $[a, b]$, $(a, b]$, $[a, b)$, and (a, b), where $a < b$, are also uncountable.

Intuition: you can list the elements of a countable set, but not of an uncountable set.

- We'll need this distinction later for discrete vs continuous probability models.
- These examples are usually sufficient for this class, but the next slide will present a formal definition.

Definition: countable and uncountable

Set A is called *countable* if one of the following is true:

1. if A is a finite set, i.e., if A has finitely many elements; or
2. if A can be put in one-to-one correspondence with natural numbers $\mathbb{N}$, in which case the set is said to be *countably infinite*.

A set is called *uncountable* if it is not countable, in which case the set is said to be *uncountably infinite*.

Again, for this class, just remember the intuition and examples from the previous slide.

Solved problems

For more solved problems, see:
`https://www.probabilitycourse.com/chapter1/1_2_5_solved1.php`

Section 1: Basics in probability theory

Section 1.3: Random Experiments

Random Experiment

A *random experiment* is a process by which we observe something uncertain.

- An *outcome* is a result of a random experiment.
- The set of all possible outcomes is called the *sample space* (which, in this context, is our universal set).

Example random experiment	Sample space
toss a coin	{heads, tails} or $\{H, T\}$
roll a 4-sided die	$\{1, 2, 3, 4, \cancel{5, 6}\}$
observe the number of goals in a soccer match	$\{0, 1, 2, 3, \ldots\}$
toss a two-headed coin	{heads} or $\{H\}$
throw a dart at a dartboard	the entire dartboard region

When we repeat a random experiment several times, we call each one of them a *trial*.

- Thus each trial has its own outcome.

Greek letter "Omega"

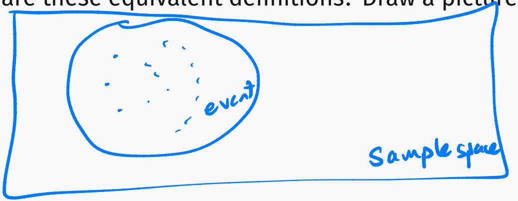
Example: toss a coin two times and observe the sequence of heads/tails. The sample space can then be written as:

If $\Omega = \{H, T\}$,

$$\{(H,H),(H,T),(T,H),(T,T)\} = \Omega \times \Omega$$

We can write this sample space as a Cartesian product of the original sample space with itself:

An *event* is a set of possible outcomes. Equivalently, it is a subset of the sample space. (Why are these equivalent definitions? Draw a picture.)



If the result of our random experiment belongs to an event, we say that *the event has occurred*.

- Six-sided die example: if we roll a 2, then the event $\{2, 4, 6\}$ has occurred, and also the event $\{3, 6\}$ did not occur.
- For events A and B, their union $A \cup B$ and intersection $A \cap B$ are also events. (Why?)
- We observe that $A \cup B$ occurs if A or B occurs.
- We observe that $A \cap B$ occurs if both A and B occurs.

Our goal in the next section will be to assign probability to certain events.