Section 4: Basics in probability theory

STA 141A – Fundamentals of Statistical Data Science

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Overview

1. Probability measure and random variables
2. PMF/PDF
3. Some distributions
4. Expected value
5. Variance and covariance
6. Conditional probability and independence

The prerequisite for this class is either STA 108 (regression) or STA 106 (ANOVA), so I expect you have already learned everything in this slide deck.

- If you need a refresher on probability, you can refer to this free textbook: `https://www.probabilitycourse.com/`

Probability measure and random variables

Probability measure - Motivation

Probability is a way to quantify randomness and/or uncertainty.

- e.g., coin flips, dice rolls, stocks, weather.
- Rules of probability should be intuitive and self-consistent.
- Self-consistent: the rules shouldn't lead to contradictions.
- Thus these rules must be constructed in a certain way.
- Suppose we want to assign a probability to each event in a set of possible events.
- We would like, at the very least:
    1. each probability to be a value between 0 and 1 (inclusive)
    2. the probability assigned to the full set of events to be 1
    3. the probability assigned to the empty set to be 0
- We need more restrictions to ensure self-consistency.

The following definition will lead to intuitive and self-consistent rules of probability.

Probability measure - Definition

Definition 1: Probabilty measure $P(\cdot)$

For a nonempty set Ω, the set function $P\colon \Omega \to [0, 1]$ is a *probability measure*, if

- $P(\Omega) = 1$,
- for any pairwise disjoints sets $A_1, A_2, \dots \subseteq \Omega$ (i.e. $A_i \cap A_j = \varnothing$ for all i, j with $i \neq j$), holds:

$$P\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} P(A_i). \tag{1}$$

This definition fulfills the three properties from the previous slide:

- $P(\Omega) = 1$: the probability of the biggest possible set is equal to 1.
- Property (1) allows us to add probabilities of disjoint sets.
  - Disjoint means having no shared elements.
  - (Property (1) is called the *countable additivity* property.)

Probability measure - Properties

Definition 1 implies the following additional properties:

Properties of $P(\cdot)$

With $\varnothing$ being the empty set, with some sets $A, B \subset \Omega$, and with $A^c = \Omega \backslash A$ denoting the complement of A, holds,

i) $P(\varnothing) = 0$;

ii) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \varnothing$;

iii) $P(A^c) = 1 - P(A)$;

iv) $P(B \setminus A) = P(B) - P(A)$ if $A \subseteq B$;

v) $P(A) \leq P(B)$ if $A \subseteq B$.

Random variables - Notion

Probability measures allow us to characterize the "randomness" of events.

- But we are often interested in more than just probabilities. For example:
  - the number of heads from three (independent) flips of some coin
  - the sum of the faces after throwing two dice
  - the lifetime of a battery
- We call each of these a *random variable* because they take on different values based on random events.
- The probability that a random variable is a certain value will depend on the probabilities of individual events.

PMF/PDF

Motivation

When doing probability calculations, rather than use probability measures (which are functions of sets), it is often easier to describe a probability distribution using functions of single variables

1. PMF/PDF

The idea behind a PMF/PDF is to assign probabilities to the possible values of a random variable.

- The concept is different for discrete and continuous random variables.

PMF/PDF - discrete and continuous case

A random variable X is *discrete* if its range is finite or countably infinite.

- Examples:
  1. number of heads after two coin flips,
  2. number of coin flips needed before a heads turns up.
- Here probabilities can be assigned to each realizable value. Examples:
  1. For $\{0, 1, 2\}$ (finite), we can assign probabilities $1/4$, $1/2$, and $1/4$.
  2. For $\mathbb{N}$ (countably infinite), we can assign probabilities $(1/2)^k$ to each $k \in \mathbb{N}$.
- The *probability mass function* (PMF) f_X of a discrete random variable X assigns probabilities to each realizable value of X. Examples:
  1. $f_X(0) = 1/4$, $f_X(1) = 1/2$, and $f_X(2) = 1/4$.
  2. $f_X(k) = (1/2)^k$ for each $k \in \mathbb{N}$.

  Here $f_X(a)$ is "the probability that X equals a."
- The probability $P(X \in A)$ that X lies in a set A can be calculated by

$$P(X \in A) = \sum_{a \in A} f_X(a), \qquad \text{with} \qquad f_X(a) := P(X = a). \tag{2}$$

- It is common to plot the PMF.

PMF/PDF - discrete and continuous case

A random variable X is *continuous* if its range is uncountably infinite.

- Examples: the lifetime of a battery, the lifetime of a person, the time it takes you to finish the first midterm exam
- For any value in the range of a continuous random variable X, the probability that X is that value must be zero. Why?
  - If uncountably many values are assigned positive probability, the sum of those values would then be infinity!
- For a continuous random variable X, at any value a we have $P(X = a) = 0$.
- The *probability density function* (PDF) f_X of a continuous random variable X describes how likely it is for X to lie a set A of values:

$$P(X \in A) = \int_A f_X(s)\mathrm{d}s. \tag{3}$$

- It is common to plot the PDF.

PMF/PDF - discrete and continuous case

From the properties of probability measures, it follows that any PMF f_X of a discrete random variable X must satisfy both

1. $f_X(x) \geq 0$ for all x, and
2. $\sum_{\text{all } x} f_X(x) = 1$.

Similarly, it follows that any PDF f_X of a continuous random variable X must satisfy both

1. $f_X(x) \geq 0$ for all x, and
2. $\int_{\text{all } x} f_X(x)\, \mathrm{d}x = 1$.

Some distributions

Discrete case - Uniform distribution

A random variable X with values in a finite set M is *uniformly* distributed if each element in M has the same probability:

$$P(X = k) = \frac{1}{\#M} \quad \text{for all } k \in M$$

- Such distributions occur when all possible outcomes are equally likely.
- We write $X \sim U(M)$ or $X \sim Unif(M)$.
- Nine random draws in R:

```r
sample(c(1,2,3,4,5,6), size=9, replace=TRUE)
```

Discrete case - Bernoulli distribution

A random variable X is *Bernoulli* distributed with parameter $p \in (0, 1)$, if $P(X = 1) = p$ and $P(X = 0) = 1 - p$.

- For when our random experiment has only two possible outcomes ("success" and "failure").
- Example: flip a coin with probability p of heads ("success"). Is it heads?
- We write $X \sim Ber_p$ or $X \sim Bern(p)$.
- Nine random draws in R: `rbinom(n=9, size=1, prob=1/3)`

Discrete case - Binomial distribution

A random variable X is *Binomial* distributed with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ if

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for all } k = 0, \dots, n.$$

- We think of n as the number of experiments and p the success probability. In the above equation, k is the number of successes.
- For measuring the probability of the number of successes of n independent Bernoulli experiments with parameter p.
- Example: flip a coin n times, each flip with probability p of heads ("success"). How many heads?
- We write $X \sim Bin_{n,p}$ or $X \sim Bin(n, p)$.
- A random draw in R: `rbinom(n=3, size=1, prob=0.25) |> sum()`

Continuous case - Uniform distribution

A random variable X is *uniformly* distributed on an interval $M = (a, b)$, with $b > a$, if the PDF has the form

$$f_X(x) = \frac{1}{b - a} \quad \text{for all } x \in (a, b).$$

- Such distributions occur when all (uncountably many) possible outcomes are equally likely.
- The interval M can also instead be $[a, b)$, or $(a, b]$, or $[a, b]$.
- Here we also write $X \sim U(M)$ or $X \sim Unif(M)$.
- Nine random draws in $(3, 5)$ in R: `runif(n=9, min=3, max=5)`

Continuous case - Normal distrobution

A random variable X is *normally* distributed with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, if the PDF has the form

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for all } x \in \mathbb{R}.$$

- This distribution appears often in this class, in future classes, and in life!
- We write $X \sim N(\mu, \sigma^2)$. We also call it *Gaussian* distributed.
- Thereby, $E(X) = \mu$ (location parameter), and $Var(X) = \sigma^2$ (squared scale).
- If $X \sim N(0, 1)$, the distribution of X is said to be *standard normal*.
- Nine random draws in R: `rnorm(n=9, mean=2, sd=1)`

PDF of $X \sim N(0, 1)$, $Y \sim N(2, 1)$, $Z \sim N(0, 3)$

Expected value

Expected value - Introduction

The expected value of a random variable is the weighted average of all of its values, where the weights are the probabilities that these values occur.

Definition 2: Expected value $E(\cdot)$

Let X be a random variable. Then, the *expected value* of X is in the discrete case and in the continuous case (given the PDF f_X) is defined as

$$E(X) = \sum_{\text{all } k} P(X = k) \cdot k \qquad \text{resp.} \qquad E(X) = \int_{\text{all } s} f_X(s) \cdot s \, \mathrm{d}s \,. \tag{4}$$

- The expected value of a random variable sometimes does not exist if, for example, the random variable is continuous and the weights are "large" for large values of the random variable (e.g. $E(X) = \int_1^\infty \frac{1}{s^2} \cdot s \mathrm{d}s = \infty$).

Expected value - Calculating expected value by hand

Calculate $E(X)$ **with PDF** $f_Y(a) = \frac{3}{7}a^2$ **where** $a \in [1, 2]$

Expected value - Calculation tools

Properties of $E(\cdot)$

Let $c \in \mathbb{R}$ be a constant, and let X, Y be random variables for which their expected values $E(X)$ and $E(Y)$ exists. Then, the following rules hold.

i) $E(c) = c$;

ii) $E(cX) = cE(X)$;

iii) $E(X + Y) = E(X) + E(Y)$.

Example with $c = 2, E(X) = 1, E(Y) = 5$

Variance and covariance

Heuristics

The variance of a random variable is the expected squared deviation of its values to its expected value.

Definition 3: Variance $Var(\cdot)$

Let X be a random variable with $E(X^2) < \infty$. Then the *variance* of X is defined as

$$Var(X) := E[\{X - E(X)\}^2]. \tag{5}$$

Think of $Var(X)$ as “how much X varies about its mean.” We can deduce:

- $Var(X) \geq 0$.
- $Var(X) = 0 \Rightarrow X$ is constant.
- The variance of X can also be calculated as

$$Var(X) = E(X^2) - (E(X))^2. \tag{6}$$

Variance - Calculation tools

Properties of $Var(\cdot)$

Let $c \in \mathbb{R}$ be a constant, and let X be a random variable with $E(X^2) < \infty$. Then

i) $Var(c) = 0$;

ii) $Var(X + c) = Var(X)$;

iii) $Var(cX) = c^2 Var(X)$;

Recall intuition: $Var(X)$ is “how much X varies about its mean.”

Example with $c = 5$, $Var(X) = 1$, $Var(Y) = 2$**.**

Covariance and correlation - Motivation

Expected value and variance help characterize the distribution of a single random variable X.

Now suppose we want to characterize the relationship between two random variables X and Y.

- A complete characterization requires assigning probabilities to every possible pair of values that (X, Y) could be.
- Simpler characterizations are the *covariance* and *correlation* of X and Y.

Heuristics

Definition 4: Covariance $Cov(\cdot, \cdot)$

Let X, Y be random variables with $E(X^2), E(Y^2) < \infty$. Then the *covariance* between X and Y is defined as

$$Cov(X, Y) := E((X - E(X))(Y - E(Y))) \,. \tag{7}$$

- The covariance between X and Y can also be calculated as

$$Cov(X, Y) = E(XY) - E(X)E(Y) \,. \tag{8}$$

- We say X and Y are *uncorrelated* if $Cov(X, Y) = 0$. Then X and Y have no linear relationship, and $E(XY) = E(X)E(Y)$.
- $Cov(X, Y) > 0$ indicate a positive linear relationship between X and Y.
- $Cov(X, Y) < 0$ indicate a negative linear relationship between X and Y.
- Covariance is symmetric: $Cov(X, Y) = Cov(Y, X)$.

Correlation coefficient

Definition 5: Correlation coefficient $\rho(\cdot, \cdot)$

Let X, Y be random variables with $E(X^2), E(Y^2) < \infty$. Then, the *correlation coefficient* between X and Y is defined as, provided $Var(X) > 0$ and $Var(Y) > 0$,

$$\rho(X, Y) := \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \in [-1, 1]. \tag{9}$$

- $\rho(X, Y) = 0 \Rightarrow$ between X and Y is no linear relationship.
- $\rho(X, Y) = -1$ (1) $\Rightarrow$ all values of X and Y lie on a line with negative (positive) slope.
- If $\rho(X, Y)$ is close to -1 (1), there is a strong negative (positive) linear relationship between X and Y.

Variance and covariance - More calculation tools

Properties of $Var(\cdot)$ and $Cov(\cdot,\cdot)$

Let $c \in \mathbb{R}$ be a constant, and let X, Y, Z be random variables with $E(X^2) < \infty$, $E(Y^2) < \infty$, and $E(Z^2) < \infty$. Then

iv) $Var(X) = Cov(X, X)$

v) $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

vi) $Cov(X, Y) = Cov(Y, X)$

vii) $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$ and $Cov(cX, Z) = cCov(X, Z)$

(Property vii says $Cov(\cdot,\cdot)$ is linear in its first argument. Because $Cov(\cdot,\cdot)$ is symmetric, it is also linear in its second argument. Thus we call it *bilinear*.)

Example with $c = 5$, $Var(X) = 1$, $Var(Y) = 2$, $Cov(X, Y) = 1/3$.

Conditional probability and independence

Heuristics

Definition and properties

An *event* is a subset of the sample space Ω.

Definition 6: Conditional probability

For events $A, B \subseteq \Omega$, the *conditional probability* of A given B is defined by

$$P(A|B) = \begin{cases} \frac{P(A \cap B)}{P(B)}, & \text{if } P(B) > 0, \\ 0, & \text{if } P(B) = 0. \end{cases} \tag{10}$$

- Events A and B are called *independent* if

  $$P(A \cap B) = P(A)P(B). \tag{11}$$

  Here knowing B provides no information about A, and vice versa.
- Equivalently, events A and B are independent if $P(A|B) = P(A)$.
- Random variables X and Y are called *independent* if for all sets A, B holds,

  $$P(X \in A, Y \in B) = P(X \in A)P(Y \in B). \tag{12}$$

- Independent random variables are uncorrelated.
- But uncorrelated random variables are not necessarily independent!